Tutorial 1

1. **Combinatorial games.**

Recall that a game is called a combinatorial game if it satisfies the following axioms.

(i) There are 2 players.

(ii) There are finite many possible positions.

(iii) The players take turns to make moves.

(iv) Perfect information, i.e. both players know the rules and the possible moves of the other player.

(v) The game ends in a finite number of moves without a draw.

A combinatorial game is said to be *impartial* if at any position, both players have the same possible moves.

Example 1. *Chess (if draws are not allowed) is a combinatorial game but it is not impartial.*

From now on, we only consider impartial combinatorial games.

Example 2. *The following games are (impartial) combinatorial games.*

(i) One-pile take away game.

(ii) Two-pile take away game.

(iii) Subtraction.

(iv) Nim.

(v) Nimber.

Example 3. *The following games are not combinatorial games.*

(i) Poker.

(ii) Rock-paper-scissors.

(iii) Tic-tac-toe.

Note. *In poker, there is no perfect information. The second game does not satisfy axiom (iii). In a game of tic-tac-toe, it is possible to get a draw.*

2. **Solving combinatorial games.**

Winning strategy: a strategy of a player that guarantees a win.

The following result is our starting point of studying combinatorial games.

Theorem 1. (Zermelo). *In any combinatorial game, exactly one of the players has a winning strategy.*

Now we may understand the problem of *solving a combinatorial game* as determining which player has a winning strategy at a given position. To do this, we need to introduce two core concepts.

N-positions and P-positions.

N-position: a position at which the player who is about to move has a winning strategy.

P-position: a position at which the player who just moved has a winning strategy.

Characterization of N-positions and P-positions:

(i) All terminal positions are P-positions.

(ii) From every P-position, any move can only reach an N-position.

(iii) From every N-position, there exists at least one move to a P-position.

Note. *1. By the above characterization, the player who reaches a P-position has a winning strategy, i.e. keeping staying in P-positions. We call a move form an N-position to a P-position a winning move.*

2. The above characterization also provides an algorithm to find all P-positions, i.e. backwards induction.

Exercise 1. *(Subtraction game). There is a pile of chips on the table. In each turn, a player removes* 2 *or* 3 *chips. The game ends when there are no possible moves and the player who makes the last move wins.*

(i) Find all P-positions.

(ii) Find all winning moves from the position that there are 2019 *chips.*

Solution: (i). The set of terminal position is $\{0, 1\}$. By the characterization of N-positions and P-positions, we have by backwards induction,

0	1	2	3	4	5	6	7	8	9	10	$11\cdots$
P	P	N	N	N	P	P	N	N	N	P	$P\cdots$

Let

$$\mathcal{P} = \{k \in \mathbb{N} : k \equiv 0 \text{ or } 1(\text{mod}5)\}.$$

We claim that $\mathcal{P}$ is the set of all P-positions. Proof of the claim: (1). Clearly the terminal positions 0 and 1 are in $\mathcal{P}$. (2). If $k \in \mathcal{P}$, then we have $k-2 \equiv 3 \text{ or } 4(\text{mod}5)$ and $k-3 \equiv 2 \text{ or } 3(\text{mod}5)$, hence $k-2 \notin \mathcal{P}$ and $k-3 \notin \mathcal{P}$. (3). If $k \notin \mathcal{P}$, then $k \equiv 2(\text{mod}5)$ or $k \equiv 3(\text{mod}5)$ or $k \equiv 4(\text{mod}5)$, in each case, we have either $k-2 \equiv 0 \text{ or } 1(\text{mod}5)$ or $k-3 \equiv 0 \text{ or } 1(\text{mod}5)$, hence either $k-2 \in \mathcal{P}$ or $k-3 \in \mathcal{P}$. By the characterization of P-positions, we complete the proof.

(ii). Clearly $2019 \equiv 4(\text{mod}5)$ is an N-position and the only winning move from 2019 is removing 3 chips to reach 2016.

3. **Nim.**

Model: suppose there are n piles of chips on the table. A move consists of choosing one pile and removing arbitrary positive number of chips from this pile. The player who makes the last move wins.

Let $\mathcal{P}$ denote the set of P-positions.

Theorem 2. $\mathcal{P} = \{(x_1, \cdots, x_n) : x_1 \oplus \cdots \oplus x_n = 0\}$.

Exercise 2. *Consider* 3*-pile nim. Find all winning moves from position* $(9, 8, 7)$*.*

Solution: Note that

$$9 \oplus 8 \oplus 7 = \begin{array}{l} (1,0,0,1)_2 \\ (1,0,0,0)_2 \\ (0,1,1,1)_2 \\ \hline (0,1,1,0)_2 \quad =6 \end{array} .$$

Hence $(9, 8, 7)$ is an N-position. The only wining move is $(0, 1, 1, 1)_2 \to (0, 0, 0, 1)_2 = 1$. In other words, the only winning move is removing 6 chips from the third pile.

There are combinatorial games which can be reduced to the nim games. The following is such an example.

Example 4. *(Nimble) Suppose there is a line of squares labeled* $0, 1, 2, \cdots$*. A finite number of coins are placed on the squares with possibly more than one coin on a single square. See Figure 1 for an example. A move consists of taking one of the coins and moving it to any square to the left. The player who makes the last move wins.*

Analysis: Note that if we view each coin in the k-th square as a pile of k chips. Then moving a coin from the k-th square to the j-th square $(j < k)$ is equivalent to removing $k - j$ chips from the pile of k chips that this coin

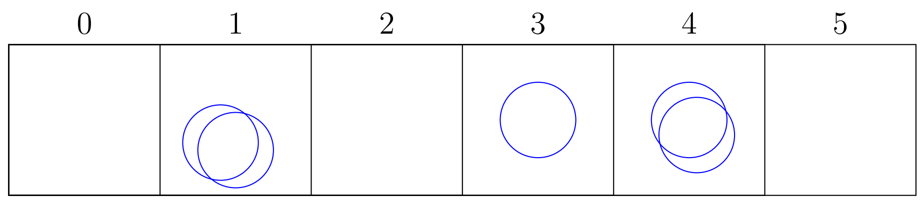

Figure 1

represents. Hence we can reduce nimble to nim. For example, the nimble in Figure 1 is equivalent to a 5-pile nim with position $(1, 1, 3, 4, 4)$.

Exercise 3. *Consider the nimble in Figure 1. Is the position a P-position? If it is not a P-position, find all winning moves.*

Solution: By the above analysis, we know that the nimble in Figure is equivalent to the 5-pile nim $(1, 1, 3, 4, 4)$. Note that

$$1 \oplus 1 \oplus 3 \oplus 4 \oplus 4 = \begin{array}{ll} (0,0,1)_2 & \\ (0,0,1)_2 & \\ (0,1,1)_2 & \\ (1,0,0)_2 & \\ (1,0,0)_2 & \\ \hline (0,1,1)_2 & =3 \end{array} .$$

Hence the position is an N-position. It is easy to see the only winning move from this position is moving the coin in the square 3 to the square 0.